Perturbative calculation of the adiabatic geometric phase and particle in a well with moving walls

1999 J. Phys. A: Math. Gen. 32 8325

(http://iopscience.iop.org/0305-4470/32/47/311)

J. Phys. A: Math. Gen. **32** (1999) 8325–8340. Printed in the UK PII: S0305-4470(99)07064-X

Perturbative calculation of the adiabatic geometric phase and particle in a well with moving walls

Ali Mostafazadeh
Department of Mathematics, Koç University, Istinye 80860, Istanbul, Turkey
E-mail: amostafazadeh@ku.edu.tr

Received 18 August 1999, in final form 5 October 1999

Abstract. We use the Rayleigh–Schrödinger perturbation theory to calculate the corrections to the adiabatic geometric phase due to a perturbation of the Hamiltonian. We show that these corrections are at least of second order in the perturbation parameter. As an application of our general results we address the problem of the adiabatic geometric phase for a one-dimensional particle which is confined to an infinite square well with moving walls.

1. Introduction

In [1], Pereshogin and Pronin consider the problem of the calculation of the adiabatic geometric phase [2] for a free particle confined between moving walls. The quantum dynamics of this system has been studied by Doescher and Rice [3], Munier *et al* [4], Berry and Klein [5], Greenbereger [6], Pinder [7], Seba [8], Makowski and co-workers [9], Devoto and Pomorisac [10] and Dodonov *et al* [11]. The analysis of Pereshogin and Pronin [1] is, however, different in nature, as it uses the geometric ideas of parallel transportation in vector bundles to derive an effective Hamiltonian for the system. They suggest that the quantum dynamics of the system is determined by the effective Hamiltonian and employ the Rayleigh–Schrödinger perturbation theory to obtain the first non-vanishing contribution to Berry's connection 1-form (vector potential) for the effective Hamiltonian.

The phenomenon of the geometric phase induced by moving boundaries was initially considered by Levy-Leblond [12], who calculated the phase shift of the wavefunction of a free particle which is forced to pass through a waveguide of finite length. There is also a mention of a 'geometric phase' in Greenberger's analysis of the dynamics of a particle confined between moving walls [6]. Greenberger's terminology is, however, not appropriate, for what he calls a geometric phase depends on spatial coordinates. Therefore, it is not really the phase of the state vector in the Hilbert space and must not be confused with the geometric phase of Berry [2] and its non-adiabatic generalization due to Aharonov and Anandan [13]. In fact, to the best of the author's knowledge, Pereshogin and Pronin's paper [1] is the only publication in which the authors use Berry's framework to study the problem of the adiabatic geometric phase due to moving boundaries.

In the present paper we address the problem of the perturbative calculation of Berry's connection 1-form for a general non-degenerate Hamiltonian. Furthermore, we use the method of time-dependent quantum canonical transformations [14, 15] to study the dynamics of a particle confined between moving walls. We then apply our general results to obtain a perturbative expression for the adiabatic geometric phase for this system. In particular, we shall consider the special case where the particle is free and compare our results with those of Pereshogin and Pronin.

This paper is organized as follows. In sections 2 and 3, we shall offer a brief review of the (cyclic and non-cyclic) adiabatic geometric phases and the Rayleigh–Schrödinger perturbation theory, respectively. In section 4, we derive an expression for Berry's connection 1-form which yields the perturbative corrections to the connection 1-form for the non-perturbed system to arbitrary orders of perturbation. In section 5, we treat the quantum dynamics of a particle confined between moving walls. In section 6, we address the problem of the adiabatic geometric phase for this system. In section 7, we summarize our main results and conclude the paper with our final remarks.

2. Adiabatic geometric phase

Consider a parametric Hamiltonian $H[R]$ satisfying the following conditions:

- $H[R]$ depends on a set of real parameters $R = (R^1, R^2, \ldots, R^d)$ which are identified with local coordinates of a smooth parameter manifold†.
- $H[R]$ is a Hermitian operator with a discrete spectrum for all possible values of R.
- The eigenvalues $E_n[R]$ of $H[R]$ are non-degenerate for all possible values of R. In particular, as R changes in time, no level crossings occur.
- The eigenvalues $E_n[R]$ of $H[R]$ are smooth functions of R.

Now if the parameters R change in time $t \in [0, \tau]$ in such a way that the evolution of the system is adiabatic [16, 17], then a normalized eigenvector $|n; R(0)\rangle$ of the initial Hamiltonian $H[R(0)]$ evolves according to [2]

$$|\psi(t)\rangle = \mathrm{e}^{\mathrm{i}\alpha_n(t)}|n; R(t)\rangle \tag{1}$$

where $\alpha_n(t)$ is a phase angle and $|n; R\rangle$ is a normalized eigenvector of $H[R]$ corresponding to the eigenvalue $E_n[R]$, i.e. $|n; R\rangle$ is a solution of

$$H[R]|n; R\rangle = E_n[R]|n; R\rangle. \tag{2}$$

We shall assume that $|n; R\rangle$ are smooth functions of R and that they form a complete orthonormal set of basis vectors for the Hilbert space. This means that for all possible values of R, m and n,

$$\langle m; R|n; R\rangle = \delta_{mn} \qquad \text{and} \qquad \sum_n |n; R\rangle\langle n; R| = 1. \tag{3}$$

The phase angle $\alpha_n(t)$ appearing in (1) is given by

$$\alpha_n(t) := \delta_n(t) + \gamma_n(t) \tag{4}$$

where

$$\delta_n(t) := -\frac{1}{\hbar}\int_0^t E_n(t')\,\mathrm{d}t' \tag{5}$$

$$\gamma_n(t) := \int_0^t \mathcal{A}_n(t')\,\mathrm{d}t' = \int_{R(0)}^{R(t)} A_n[R] \tag{6}$$

$$\mathcal{A}_n(t) := \mathrm{i}\langle n; R(t)|\frac{\mathrm{d}}{\mathrm{d}t}|n; R(t)\rangle \tag{7}$$

† Here R denotes $(R^1, R^2, \ldots, R^d)$. This notation does not mean that R is a vector belonging to $\mathbb{R}^d$. R is a d-tuple of real numbers representing the coordinates of a smooth parameter manifold. The latter must not be confused with the configuration space of the corresponding system.

and

$$A_n[R] := \mathrm{i}\langle n;\, R|d|n;\, R\rangle = \sum_{a=1}^{d} i\,\langle n;\, R|\frac{\partial}{\partial R^a}|n;\, R\rangle\, \mathrm{d}R^a. \tag{8}$$

$\delta_n(t)$ and $\gamma_n(t)$ are called the *dynamical* and *geometrical* parts of the *total phase angle* $\alpha_n(t)$, respectively. The 1-form $A_n[R]$ is known as *Berry's connection 1-form* [2].

The adiabatic geometric phase [18, 19] is given by

$$\Phi_n(t) = W_n(t)\,\Gamma^n(t) \tag{9}$$

where

$$\Gamma^n(t) := \mathrm{e}^{\mathrm{i}\gamma_n(t)} \tag{10}$$

and

$$W_n(t) := \langle n;\, R(0)|n;\, R(t)\rangle. \tag{11}$$

If the parameters $R(t)$ trace a closed path C in the parameter space, i.e. there is $T \in \mathbb{R}^+$ such that $R(T) = R(0)$, then at $t = T$ we have $H[R(T)] = H[R(0)]$, $|n;\, R(T)\rangle = |n;\, R(0)\rangle$ and $W_n(T) = 1$. In particular, $|\psi(0)\rangle = |n;\, R(0)\rangle$ undergoes a cyclic evolution and $\Phi_n(T)$ yields the cyclic adiabatic geometric phase or Berry's phase [2]:

$$\Phi_n(T) = \Gamma^n(T) = \mathrm{e}^{\mathrm{i}\gamma_n(T)} = \mathrm{e}^{\mathrm{i}\oint_C A_n[R]}. \tag{12}$$

The above derivation of the geometric phase is valid even for the cases where the Hilbert space is time dependent. The time dependence of the Hilbert space may be reflected in the definition of the measure used to define integration. For example, consider the problem of a one-dimensional particle confined between two walls positioned as $x = 0$ and $L(t)$, where $L : [0, \tau] \to \mathbb{R}^+$ is a smooth function and τ is the duration of the evolution of the system. The Hilbert space $\mathcal{H}_t$ is $L^2([0, L(t)])$ which depends on time. However, we can identify $L^2([0, L(t)])$ with

$$L^2_\mu(\mathbb{R}) := \left\{\psi : \mathbb{R} \to \mathbb{C}\, \middle|\, \int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,\mu_t(x)\,\mathrm{d}x < \infty\right\} \tag{13}$$

where the measure function μ_t is given by

$$\mu_t(x) := \theta(x) - \theta(x - L(t)) \tag{14}$$

and $\theta : \mathbb{R} \to \{0, 1\}$ denotes the step function

$$\theta(x) := \begin{cases} 1 & \text{for} \quad x \geqslant 0 \\ 0 & \text{for} \quad x < 0. \end{cases} \tag{15}$$

Now let us denote the eigenfunctions of $H[R(t)]$ in the position representation by ϕ_n, i.e. $\phi_n(x;\, t) := \langle x|n;\, R(t)\rangle$. Then

$$\begin{aligned} \mathcal{A}_n^* &= -\mathrm{i}\left(\langle n;\, R(t)|\frac{\mathrm{d}}{\mathrm{d}t}|n;\, R(t)\rangle\right)^* = -\mathrm{i}\left(\int_{-\infty}^{\infty} \phi_n^*\dot{\phi}_n\mu\,\mathrm{d}x\right)^* = -\mathrm{i}\int_{-\infty}^{\infty} \phi_n\dot{\phi}_n^*\mu\,\mathrm{d}x \\ &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\int_{-\infty}^{\infty} \phi_n\phi_n^*\mu\,\mathrm{d}x + \mathrm{i}\int_{-\infty}^{\infty} \phi_n^*\dot{\phi}_n\mu\,\mathrm{d}x + \mathrm{i}\int_{-\infty}^{\infty} |\phi_n|^2\dot{\mu}\,\mathrm{d}x \\ &= \mathcal{A}_n + \mathrm{i}|\phi_n(x = L(t);\, t)|^2 \end{aligned} \tag{16}$$

where * stands for the operation of complex conjugation. The last equation in (16) is obtained by making use of the fact that ϕ_n are normalized and that $\dot{\mu} = -\dot{\theta}(x - L(t)) = \delta(x - L(t))$ where $\delta(x)$ is the Dirac delta function. Equation (16) shows that $\mathcal{A}_n$ and consequently the Berry connection 1-form A_n and the phase angles $\gamma_n(t)$ and $\alpha_n(t)$ are real, provided that one chooses the boundary condition $\phi_n|_{x=L(t)} = 0$.

3. Perturbation theory

In order to compute the adiabatic geometric phase (9), one needs to obtain the eigenvectors $|n; R\rangle$ of the Hamiltonian $H[R]$. There are, however, quite a few Hamiltonians whose eigenvalue equation is solved exactly. Often, one uses approximation schemes to obtain the eigenvalues and eigenvectors of a given Hamiltonian. One of the best known approximation methods of solving the eigenvalue problem is Rayleigh–Schrödinger perturbation theory [20, 21]. Next, we shall derive the basic results of Rayleigh–Schrödinger perturbation theory.

Consider a parametric Hamiltonian of the form

$$H[R] = H_0[R] + \epsilon[R]\, h[R] \tag{17}$$

where $H_0[R]$ is a parametric Hamiltonian with the same properties as $H[R]$, $\epsilon[R]$ is a real parameter and $h[R]$ is a Hermitian operator. If the eigenvalue equation for $H_0[R]$ is exactly solvable, then one may attempt to obtain the eigenvalues and eigenvectors of $H[R]$ as power series in $\epsilon[R]$,

$$E_n[R] = \sum_{\ell=0}^{\infty} E_n^{(\ell)}[R]\, \epsilon[R]^{\ell} \tag{18}$$

$$|n; R\rangle = \sum_{\ell=0}^{\infty} |n; R\rangle_{\ell}\, \epsilon[R]^{\ell} \tag{19}$$

whose coefficients $E_n^{(\ell)}[R]$ and $|n; R\rangle_{\ell}$ are expressed in terms of the eigenvalues and eigenvectors of $H_0[R]$.

In the following calculations we shall suppress the R dependence of the relevant quantities for brevity, i.e. we shall use the notation

$$H = H[R] \qquad H_0 = H_0[R] \qquad \epsilon = \epsilon[R] \qquad h = h[R]$$
$$E_n = E_n[R] \qquad |n\rangle = |n; R\rangle \qquad E_n^{(\ell)} = E_n^{(\ell)}[R] \qquad |n\rangle_{\ell} = |n; R\rangle_{\ell}.$$

Substituting equations (18) and (19) into equation (2) and performing the necessary calculations, we obtain an equation of the form

$$\sum_{\ell=0}^{\infty} |\xi\rangle_{\ell}\, \epsilon^{\ell} = 0 \tag{20}$$

where

$$|\xi\rangle_0 = (H_0 - E_n^{(0)})|n\rangle_0 \tag{21}$$

and

$$|\xi\rangle_{\ell} = H_0|n\rangle_{\ell} + h|n\rangle_{\ell-1} - \sum_{k=0}^{\ell} E_n^{(k)}|n\rangle_{\ell-k} \qquad \text{for} \quad \ell \geqslant 1. \tag{22}$$

The basic idea of the perturbation theory is to construct a solution of equation (20) by requiring

$$|\xi\rangle_{\ell} = 0 \qquad \text{for all} \quad \ell = 0, 1, 2, \ldots. \tag{23}$$

For $\ell = 0$, this implies that $E_n^{(0)}$ and $|n\rangle_0$ are the eigenvalues and eigenvectors of H_0. Therefore, according to the hypothesis we can calculate them exactly. We shall assume without loss of generality that $|n\rangle_0$ form a complete orthonormal set of basis vectors of the Hilbert space and express $|n\rangle_{\ell}$ in this basis. This leads to

$$|n\rangle_{\ell} = \sum_m C_{mn}^{\ell}|m\rangle_0 \tag{24}$$

where $C^{\ell}_{mn} = C^{\ell}_{mn}[R]$ are complex coefficients depending on the parameters R. Clearly, $C^{\ell}_{mn} = {}_0\langle m|n\rangle_{\ell}$. In particular,

$$C^0_{mn} = \delta_{mn}. \tag{25}$$

Now let us substitute equation (24) into equation (22) and use equation (25) and the identity

$$h = \sum_{r,s} {}_0\langle r|h|s\rangle_0 \, |r\rangle_0 \, {}_0\langle s| \tag{26}$$

to simplify the resulting expression. This yields

$$|\xi\rangle_{\ell} = \sum_m d^{\ell}_m |m\rangle_0 \qquad \text{where} \quad \ell \geqslant 1 \tag{27}$$

$$d^1_m = -\delta_{mn} E^{(1)}_n + (E^{(0)}_m - E^{(0)}_n) C^1_{mn} + {}_0\langle m|h|n\rangle_0 \tag{28}$$

$$d^{\ell}_m = -\delta_{mn} E^{(\ell)}_n + (E^{(0)}_m - E^{(0)}_n) C^{\ell}_{mn} + \sum_r C^{\ell-1}_{rn} {}_0\langle m|h|r\rangle_0 - \sum_{k=1}^{\ell-1} E^{(k)}_n C^{(\ell-k)}_{mn} \qquad \text{for } \ell \geqslant 2. \tag{29}$$

Next we enforce equation (23). In view of equation (27) and the linear independence of the basis vectors $|m\rangle_0$, equation (23) implies $d^{\ell}_m = 0$ for all m and $\ell \geqslant 1$. For $\ell = 1$, this leads to

$$E^1_n = {}_0\langle n|h|n\rangle_0 \tag{30}$$

and

$$C^1_{mn} = \frac{{}_0\langle m|h|n\rangle_0}{E^{(0)}_n - E^{(0)}_m} \qquad \text{for} \quad m \neq n. \tag{31}$$

Equations (30) and (31) are obtained by setting $m = n$ and $m \neq n$ in $d^1_m = 0$, respectively. Similarly, $d^{\ell}_m = 0$ for $\ell \geqslant 2$ give rise to

$$E^{(\ell)}_n = \sum_r C^{\ell-1}_{rn} {}_0\langle n|h|r\rangle_0 - \sum_{k=1}^{\ell-1} E^k_n C^{\ell-k}_{nn} \qquad \text{for} \quad \ell \geqslant 2 \tag{32}$$

$$C^{\ell}_{mn} = (E^{(0)}_n - E^{(0)}_m)^{-1} \left(\sum_r C^{\ell-1}_{rn} {}_0\langle m|h|r\rangle_0 - \sum_{k=1}^{\ell-1} E^{(k)}_n C^{\ell-k}_{mn} \right) \qquad \text{for} \quad m \neq n \quad \ell \geqslant 2. \tag{33}$$

We can use equations (30) and (31), to write equations (32) and (33) in the form

$$E^{(2)}_n = \sum_{r \neq n} (E^{(0)}_r - E^{(0)}_n) C^1_{nr} C^1_{rn} \tag{34}$$

$$E^{(\ell)}_n = \sum_{r \neq n} (E^{(0)}_r - E^{(0)}_n) C^1_{nr} C^{\ell-1}_{rn} - \sum_{k=2}^{\ell-1} E^{(k)}_n C^{\ell-k}_{nn} \qquad \text{for} \quad \ell \geqslant 3 \tag{35}$$

$$C^2_{mn} = \sum_{r \neq m} \left(\frac{E^{(0)}_r - E^{(0)}_m}{E^{(0)}_n - E^{(0)}_m} \right) C^1_{mr} C^1_{rn} + \left(\frac{E^{(1)}_m - E^{(1)}_n}{E^{(0)}_n - E^{(0)}_m} \right) C^1_{mn} \qquad \text{for} \quad m \neq n \tag{36}$$

$$C^{\ell}_{mn} = \sum_{r \neq m} \left(\frac{E^{(0)}_r - E^{(0)}_m}{E^{(0)}_n - E^{(0)}_m} \right) C^1_{mr} C^{\ell-1}_{rn} + \left(\frac{E^{(1)}_m - E^{(1)}_n}{E^{(0)}_n - E^{(0)}_m} \right) C^{\ell-1}_{mn} - \sum_{k=2}^{\ell-1} \frac{E^{(k)}_n C^{\ell-k}_{mn}}{E^{(0)}_n - E^{(0)}_m} \qquad \text{for} \quad m \neq n \quad \ell \geqslant 3. \tag{37}$$

These equations yield $E_n^{(\ell)}$ and C_{mn}^{ℓ} with $m \neq n$ in terms of $E_r^{(k)}$, $E_r^{(k)}$, C_{rs}^k and C_{rr}^k where $k < \ell$. One can iterate them to express $E_n^{(\ell)}$ and C_{mn}^{ℓ} with $m \neq n$ in terms of $E_r^{(0)}$, $E_r^{(1)}$, C_{rs}^1 and C_{rr}^k. They do not, however, restrict C_{nn}^{ℓ}. This means that C_{nn}^{ℓ} are not fixed by the eigenvalue equation. This is due to the fact that the eigenvalue equation (2) determines the eigenvectors up to an arbitrary multiplicative factor. We can restrict the choice of C_{nn}^{ℓ} by imposing the normalization condition on $|n\rangle$. Substituting equations (19) and (24) in $\langle m|n\rangle = \delta_{mn}$ and making use of the orthonormality of $|n\rangle_0$, we find

$$\sum_{j=1}^{\infty} d_j \epsilon^j = 0 \tag{38}$$

where

$$d_j := \sum_{\ell=0}^{j} \sum_r C_{rn}^{\ell} C_{rm}^{j-\ell *}. \tag{39}$$

Again we seek a solution of equation (38) of the form $d_j = 0$ for all $j = 1, 2, \ldots$. This leads to

$$C_{nm}^{1*} + C_{mn}^1 = 0 \tag{40}$$

$$C_{nm}^{j*} + C_{mn}^j = -\sum_{\ell=1}^{j-1} \sum_r C_{rn}^{\ell} C_{rm}^{j-\ell *} \qquad \text{for} \quad j \geqslant 2. \tag{41}$$

One can show that for $m \neq n$, equations (40) and (41) are trivially satisfied. However, for $m = n$, they determine the real part of C_{nn}^{ℓ} according to

$$\mathrm{Re}(C_{nn}^1) = 0 \tag{42}$$

$$\mathrm{Re}(C_{nn}^j) = -\tfrac{1}{2} \sum_{\ell=1}^{j-1} \sum_r C_{rn}^{\ell} C_{rn}^{j-\ell^*} \qquad \text{for} \quad j \geqslant 2 \tag{43}$$

where Re means the 'real part of'. The imaginary part of C_{nn}^{ℓ} is still arbitrary. This is because the normalization condition determines the eigenvectors up to an arbitrary phase factor. This phase factor can, in principle, depend on the perturbation parameter ϵ and consequently show up in all orders of perturbation. The common practice is to set the imaginary part of C_{nn}^{ℓ} equal to zero [20]. This corresponds to making a particular choice for the phase of the eigenvectors.

4. Perturbative calculation of Berry's connection 1-form

Having obtained the perturbation series for the eigenvectors $|n\rangle$ of the Hamiltonian H, we are in a position to compute Berry's connection 1-form A_n. In fact, we shall instead compute $\mathcal{A}_n$ of equation (7). A_n can be easily obtained from $\mathcal{A}_n$ by changing the time derivatives to the exterior derivatives.

We shall first substitute equation (24) into equation (19). This yields

$$|n\rangle = \sum_{\ell=0}^{\infty} \sum_m C_{mn}^{\ell} |m\rangle_0 \epsilon^{\ell}. \tag{44}$$

Next, we differentiate both sides of equation (44) and take the inner product of the resulting expression with $|n\rangle$. Then using equation (24), the identity

$${}_k\langle n|\frac{\mathrm{d}}{\mathrm{d}t}|m\rangle_0 = \sum_r {}_k\langle n|r\rangle_0\, {}_0\langle r|\frac{\mathrm{d}}{\mathrm{d}t}|m\rangle_0$$

and performing the necessary algebra, we find

$$\mathcal{A}_n = \mathrm{i}\langle n|\frac{\mathrm{d}}{\mathrm{d}t}|n\rangle = \mathrm{i}\sum_{\ell,k=0}^{\infty}\sum_m\left[\left(C_{mn}^{k*}\dot{C}_{mn}^{\ell} + \sum_r C_{rn}^{k*}C_{mn0}^{\ell}\langle r|\frac{\mathrm{d}}{\mathrm{d}t}|m\rangle_0\right)\epsilon^{\ell+k} + \ell C_{mn}^{k*}C_{mn}^{\ell}\dot{\epsilon}\epsilon^{\ell+k-1}\right] \tag{45}$$

where a dot denotes a time derivative.

Making the change of dummy index $k \to j := \ell + k$ we can write equation (45) in the form

$$\mathcal{A}_n = \sum_{j=0}^{\infty}\sum_{\ell=0}^{j}\sum_m\left[\mathrm{i}C_{mn}^{j-\ell*}\dot{C}_{mn}^{\ell} + \left(C_{mn}^{j-\ell*}C_{mn}^{\ell}\mathcal{A}_m^{(0)}\right)\right]\epsilon^j$$
$$+\sum_{j=0}^{\infty}\sum_{\ell=0}^{j}\sum_m\sum_{r\neq m}C_{rn}^{j-\ell*}C_{mn}^{\ell}\mathcal{A}_{rm}^{(0)}\epsilon^j + \mathrm{i}\sum_{j=1}^{\infty}\sum_{\ell=1}^{j}\sum_m \ell C_{mn}^{j-\ell*}C_{mn}^{\ell}\dot{\epsilon}\epsilon^{j-1} \tag{46}$$

where $\mathcal{A}_{rm}^{(0)} := i_0\langle r|\frac{\mathrm{d}}{\mathrm{d}t}|m\rangle_0$ and $\mathcal{A}_m^{(0)} := \mathcal{A}_{mm}^{(0)} = i_0\langle m|\frac{\mathrm{d}}{\mathrm{d}t}|m\rangle_0$.

The last term on the right-hand side of (46) may be written as

$$\ell C_{mn}^{j-\ell*}C_{mn}^{\ell}\dot{\epsilon}\epsilon^{j-1} = \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\frac{\ell}{j}\right)C_{mn}^{j-\ell*}C_{mn}^{\ell}\epsilon^j\right] - \frac{\ell}{j}\left(\dot{C}_{mn}^{j-\ell*}C_{mn}^{\ell} + C_{mn}^{j-\ell*}\dot{C}_{mn}^{\ell}\right)\epsilon^j.$$

Substituting this equation in (46), writing the $j = 0, 1$ and 2 terms in (46) separately, and making use of $\dot{C}_{mn}^0 = \dot{\delta}_{mn} = 0$, we obtain

$$\mathcal{A}_n = \mathcal{A}_n^{(0)} + \left[2\,\mathrm{Re}(C_{nn}^1)\,\mathcal{A}_n^{(0)} + 2\sum_{r\neq n}\mathrm{Re}(C_{rn}^1\mathcal{A}_{nr}^{(0)})\right]\epsilon + \left[2\,\mathrm{Re}(C_{nn}^2)\,\mathcal{A}_n^{(0)} + \sum_m |C_{mn}^1|^2\mathcal{A}_m^{(0)}\right.$$
$$+\tfrac{1}{2}\mathrm{i}\sum_m(C_{mn}^{1*}\dot{C}_{mn}^1 - \dot{C}_{mn}^{1*}C_{mn}^1) + \sum_{r\neq n}2\,\mathrm{Re}(C_{rn}^2\mathcal{A}_{nr}^{(0)}) + \left.\sum_m\sum_{r\neq m}C_{rn}^{1*}C_{mn}^1\mathcal{A}_{rm}^{(0)}\right]\epsilon^2$$
$$+\sum_{j=3}^{\infty}\sum_{\ell=0}^{j}\sum_m\left[C_{mn}^{j-\ell*}C_{mn}^{\ell}\mathcal{A}_m^{(0)} + \mathrm{i}\left(1-\frac{\ell}{j}\right)C_{mn}^{j-\ell*}\dot{C}_{mn}^{\ell} - \frac{\mathrm{i}\ell}{j}\dot{C}_{mn}^{j-\ell*}C_{mn}^{\ell}\right.$$
$$\left.+\sum_{r\neq m}C_{rn}^{j-\ell*}C_{mn}^{\ell}\mathcal{A}_{rm}^{(0)}\right]\epsilon^j + \mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}t} \tag{47}$$

where

$$f := \sum_{j=1}^{\infty}\sum_{\ell=1}^{j}\sum_m\left(\frac{\ell}{j}\right)C_{mn}^{j-\ell*}C_{mn}^{\ell}\epsilon^j. \tag{48}$$

The first term in the first square bracket on the right-hand side of equation (47) vanishes by virtue of equation (42). Similarly, using equation (43), we can write the first two terms of the second square bracket in the form

$$2\,\mathrm{Re}(C_{nn}^2)\,\mathcal{A}_n^{(0)} + \sum_m |C_{mn}^1|^2\mathcal{A}_m^{(0)} = \sum_m\left(-|C_{mn}^1|^2\mathcal{A}_n^{(0)} + |C_{mn}^1|^2\mathcal{A}_m^{(0)}\right)$$
$$= \sum_{m\neq n}|C_{mn}^1|^2\left(\mathcal{A}_m^{(0)} - \mathcal{A}_n^{(0)}\right). \tag{49}$$

Next let us observe that in view of equation (42),

$$C_{nn}^{1*}\dot{C}_{nn}^1 - C_{nn}^1\dot{C}_{nn}^{1*} = 0. \tag{50}$$

Substituting equations (42), (49) and (50) in equation (47), we obtain

$$\mathcal{A}_n = \mathcal{A}_n^{(0)} + \left[2\sum_{r\neq n}\mathrm{Re}(C^1_{rn}\mathcal{A}^{(0)}_{nr})\right]\epsilon + \left[\sum_{m\neq n}\left\{|C^1_{mn}|^2(\mathcal{A}^{(0)}_m - \mathcal{A}^{(0)}_n) + \tfrac{1}{2}\mathrm{i}(C^{1*}_{mn}\dot{C}^1_{mn} - \dot{C}^{1*}_{mn}C^1_{mn})\right.\right.$$
$$\left.\left. + 2\,\mathrm{Re}(C^2_{mn}\mathcal{A}^{(0)}_{nm})\right\} + \sum_m\sum_{r\neq m} C^{1*}_{rn}C^1_{mn}\mathcal{A}^{(0)}_{rm}\right]\epsilon^2 + \mathcal{O}(\epsilon^3) + \mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}t} \tag{51}$$

where $\mathcal{O}(\epsilon^3)$ denotes the third- and higher-order terms in ϵ, i.e.

$$\mathcal{O}(\epsilon^3) := \sum_{j=3}^{\infty}\sum_{\ell=0}^{j}\sum_m\left[C^{j-\ell*}_{mn}C^{\ell}_{mn}\mathcal{A}^{(0)}_m + \mathrm{i}\left(1-\frac{\ell}{j}\right)C^{j-\ell*}_{mn}\dot{C}^{\ell}_{mn} - \frac{\mathrm{i}\ell}{j}\dot{C}^{j-\ell*}_{mn}C^{\ell}_{mn}\right.$$
$$\left. + \sum_{r\neq m} C^{j-\ell*}_{rn}C^{\ell}_{mn}\mathcal{A}^{(0)}_{rm}\right]\epsilon^j. \tag{52}$$

We can rewrite the terms involving $\mathcal{A}^{(0)}_m$ in (52) by separating the $\ell = 0$ and j terms in the sum and using equation (43) which yields

$$\sum_{\ell=0}^{j}\sum_m C^{j-\ell*}_{mn}C^{\ell}_{mn}\mathcal{A}^{(0)}_m = 2\,\mathrm{Re}(C^j_{nn})\mathcal{A}^{(0)}_n + \sum_{\ell=1}^{j-1}\sum_m C^{j-\ell*}_{mn}C^{\ell}_{mn}\mathcal{A}^{(0)}_m$$
$$= \sum_{m\neq n}\sum_{\ell=1}^{j-1} C^{j-\ell*}_{mn}C^{\ell}_{mn}(\mathcal{A}^{(0)}_m - \mathcal{A}^{(0)}_n). \tag{53}$$

Furthermore, changing the dummy index ℓ in the second sum on the right-hand side of (52) to $k := j - \ell$, we have

$$\sum_{\ell=0}^{j} i\left(1-\frac{\ell}{j}\right)C^{j-\ell*}_{mn}\dot{C}^{\ell}_{mn} = \sum_{k=0}^{j}\frac{\mathrm{i}k}{j}C^{k*}_{mn}\dot{C}^{j-k}_{mn}. \tag{54}$$

Substituting equations (53) and (54) in (52), we find

$$\mathcal{O}(\epsilon^3) = \sum_{j=3}^{\infty}\sum_{\ell=1}^{j-1}\left[\sum_{m\neq n} C^{j-\ell*}_{mn}C^{\ell}_{mn}(\mathcal{A}^{(0)}_m - \mathcal{A}^{(0)}_n)\right.$$
$$\left. + \sum_m\left\{\frac{\mathrm{i}\ell}{j}(C^{j-\ell*}_{mn}\dot{C}^{\ell}_{mn} - \dot{C}^{j-\ell*}_{mn}C^{\ell}_{mn}) + \sum_{r\neq m} C^{j-\ell*}_{rn}C^{\ell}_{mn}\mathcal{A}^{(0)}_{rm}\right\}\right]\epsilon^j. \tag{55}$$

Having obtained the perturbation series for $\mathcal{A}_n$ we can write down the perturbation series for the Berry's connection 1-form A_n. Changing the time derivatives to the exterior derivatives in the expression (51) for $\mathcal{A}_n$, we find

$$A_n = A_n^{(0)} + \left[2\sum_{r\neq n}\mathrm{Re}(C^1_{rn}A^{(0)}_{nr})\right]\epsilon + \left[\sum_{m\neq n}\left\{|C^1_{mn}|^2(A^{(0)}_m - A^{(0)}_n) + \tfrac{1}{2}\mathrm{i}(C^{1*}_{mn}\,\mathrm{d}C^1_{mn} - \mathrm{d}C^{1*}_{mn}C^1_{mn})\right.\right.$$
$$\left.\left. + 2\,\mathrm{Re}(C^2_{mn}A^{(0)}_{nm})\right\} + \sum_m\sum_{r\neq m} C^{1*}_{rn}C^1_{mn}A^{(0)}_{rm}\right]\epsilon^2 + \mathrm{O}(\epsilon^3) + \mathrm{i}\,\mathrm{d}f \tag{56}$$

where

$$A^{(0)}_{mn} := i_0\langle m|d|n\rangle_0 \qquad A^{(0)}_n := A^{(0)}_{nn} = i_0\langle n|d|n\rangle_0 \tag{57}$$

and

$$\mathrm{O}(\epsilon^3) := \sum_{j=3}^{\infty}\sum_{\ell=1}^{j-1}\Bigg[\sum_{m\neq n} C_{mn}^{j-\ell *}C_{mn}^{\ell}\big(A_m^{(0)} - A_n^{(0)}\big) + \sum_{m}\left\{\frac{\mathrm{i}\ell}{j}\big(C_{mn}^{j-\ell *}\,\mathrm{d}C_{mn}^{\ell} - \mathrm{d}C_{mn}^{j-\ell *}\,C_{mn}^{\ell}\big) + \sum_{r\neq m} C_{rn}^{j-\ell *}C_{mn}^{\ell}A_{rm}^{(0)}\right\}\Bigg]\epsilon^j. \tag{58}$$

In particular, let us consider a case where $A_{mn}^{(0)} = 0$ for all m and n. Then,

$$A_n = \sum_{m\neq n}\tfrac{1}{2}\mathrm{i}\big(C_{mn}^{1*}\,\mathrm{d}C_{mn}^{1} - \mathrm{d}C_{mn}^{1*}\,C_{mn}^{1}\big)\epsilon^2 + \mathrm{O}(\epsilon^3) + \mathrm{i}\,\mathrm{d}f. \tag{59}$$

If the unperturbed Hamiltonian H_0 is a fixed operator, its eigenvectors $|n\rangle_0$ will not depend on R. In this case $A_{mn}^{(0)} = 0$ and equation (59) holds. This equation indicates that *the geometric phase effects due to a time-dependent perturbation are second- (or higher-) order effects in the perturbation parameter.* In fact, this statement is also valid for the general case where $A_{mn}^{(0)} \neq 0$. In order to see this, we recall the hypothesis of the adiabaticity of the evolution [17] which requires

$$\mathcal{A}_{mn} := \mathrm{i}\langle m|\frac{\mathrm{d}}{\mathrm{d}t}|n\rangle \approx 0 \qquad \text{for all} \quad m \neq n. \tag{60}$$

We can repeat the above calculation of $\mathcal{A}_n$ for $\mathcal{A}_{mn}$ with $m \neq n$ and show that

$$\mathcal{A}_{mn} = \mathcal{A}_{mn}^{(0)} + \text{terms of order } \epsilon \text{ and higher.}$$

Hence, in order to ensure the validity of the adiabaticity condition (60), $\mathcal{A}_{mn}^{(0)}$ with $m \neq n$ must be at least of order ϵ. Consequently, the first perturbative correction to Berry's connection 1-form (56) is indeed of order ϵ^2.

5. Particle in a one-dimensional infinite well with moving boundaries

The Schrödinger equation for a particle of mass M in a one-dimensional infinite square well with a moving boundary is given by

$$\mathrm{i}\hbar\dot{\psi}(x;t) = \left[-\frac{\hbar^2}{2M}\frac{\partial^2}{\partial x^2} + V(x,t)\right]\psi(x;t) \tag{61}$$

$$\psi(0;t) = \psi(L(t);t) = 0 \tag{62}$$

where $V(x,t)$ is a real interaction potential, $L : [0,\tau] \to \mathbb{R}^+$ is a smooth function, τ is the duration of the evolution of the system, and $x = 0$ and $L(t)$ are the positions of the boundaries.

As argued by Pereshogin and Pronin [1], who studied the case of a free particle ($V = 0$), the Hilbert space $\mathcal{H}_t$ of this system at time t is $L^2([0, L(t)])$. In particular, $\mathcal{H}_t$ is time dependent. One way to handle this situation is to identify $\mathcal{H}_t$ with a fibre of a vector bundle, endow this vector bundle with a connection, and replace the ordinary time derivative appearing in the Schrödinger equation (61) by the covariant time derivative corresponding to this connection. This is the approach pursued by Pereshogin and Pronin [1]. If one makes the same choice for the connection as the one made by Pereshogin and Pronin [1], then one obtains the effective Hamiltonian

$$H_{\mathrm{eff}}(t) = \frac{p^2}{2M} + \frac{\dot{L}(t)}{2L(t)}(xp + px) \tag{63}$$

which is valid for $V = 0$. Pereshogin and Pronin suggest that the dynamics of such a particle is determined by the Schrödinger equation for this effective Hamiltonian subject to the same boundary conditions as in (62)†.

The conventional approach to this problem is to determine the dynamics of the system using the Hamiltonian [3]

$$H(t) = \frac{p^2}{2M} + \tilde{V}(x; t) \tag{64}$$

where

$$\tilde{V}(x, t) = \begin{cases} V(x, t) & \text{for} \quad x \in [0, L(t)] \\ \infty & \text{for} \quad x \notin [0, L(t)]. \end{cases} \tag{65}$$

The Schrödinger equation for this Hamiltonian is clearly equivalent to the original Schrödinger equation (61).

We shall approach the problem of solving the Schrödinger equation for this system by applying the time-dependent canonical transformation [14, 15],

$$|\psi(t)\rangle \to |\psi'(t)\rangle := \mathcal{U}(t)\,|\psi(t)\rangle \tag{66}$$

$$H(t) \to H'(t) := \mathcal{U}(t)H(t)\,\mathcal{U}(t)^\dagger + \mathrm{i}\hbar\,\mathcal{U}(t)\,\dot{\mathcal{U}}(t)^\dagger \tag{67}$$

$$x \to x' := \mathcal{U}(t)\,x\,\mathcal{U}(t)^\dagger \qquad p \to p' := \mathcal{U}(t)\,p\,\mathcal{U}(t)^\dagger \tag{68}$$

defined by the unitary operator

$$\mathcal{U}(t) := \mathrm{e}^{(\mathrm{i}a(t)/2\hbar)(xp+px)} \tag{69}$$

where $a = a(t)$ is a smooth real-valued function of time. This canonical transformation corresponds to a time-dependent dilatation of space [15]. This is easily seen by substituting (69) into (68) which yields

$$x \to x' = \mathrm{e}^{a(t)}x \qquad \text{and} \qquad p \to p' = \mathrm{e}^{-a(t)}p. \tag{70}$$

Furthermore, substituting (69) in (67) and using equation (64), we find

$$H(t) \to H'(t) = \frac{p'^2}{2M} + \tilde{V}'(x, t) - \frac{\dot{a}(t)}{2}(xp + px) \tag{71}$$

where

$$\tilde{V}'(x, t) := \tilde{V}(x', t). \tag{72}$$

Next let us choose the dilatation parameter $a(t)$ to be

$$a(t) = \ln(L(t)/L_0) \tag{73}$$

where $L_0 := L(0)$. Substituting (73) in (70), we obtain

$$x' = \left(\frac{L(t)}{L_0}\right)x \qquad \text{and} \qquad p' = \left(\frac{L_0}{L(t)}\right)p. \tag{74}$$

In view of equations (65), (72) and (74), the transformed potential is given by

$$\begin{aligned} \tilde{V}'(x, t) = \tilde{V}(x', t) &:= \begin{cases} V(x', t) & \text{for} \quad x' \in [0, L(t)] \\ \infty & \text{for} \quad x' \notin [0, L(t)] \end{cases} \\ &= \begin{cases} V\left(\dfrac{L(t)x}{L_0}, t\right) & \text{for} \quad x \in [0, L_0] \\ \infty & \text{for} \quad x \notin [0, L_0]. \end{cases} \end{aligned} \tag{75}$$

† As we shall see below, a consistent treatment of this problem leads to an effective Hamiltonian which differs from H_{eff} in the sign of the second term on the right-hand side of (63).

This means that the Schrödinger equation for the transformed Hamiltonian $H'(t)$ is equivalent to the Schrödinger equation for the Hamiltonian

$$H''(t) = \frac{L_0^2 p^2}{2ML(t)^2} - \frac{\dot{L}(t)}{2L(t)}(xp + px) + V\left(\frac{L(t)x}{L_0}, t\right) \tag{76}$$

namely

$$\mathrm{i}\hbar\dot{\psi}'(x;t) = \left[-\frac{\hbar^2 L_0^2}{2ML(t)^2}\frac{\partial^2}{\partial x^2} + \frac{\mathrm{i}\hbar\dot{L}(t)}{2L(t)}\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x\right) + V\left(\frac{L(t)x}{L_0}, t\right)\right]\psi'(x;t) \tag{77}$$

where $\psi'(x;t) \in L^2([0, L_0])$ and (77) is supposed to be solved with boundary conditions

$$\psi'(0;t) = \psi'(L_0;t) = 0. \tag{78}$$

The canonical transformation defined by (69) and (73), therefore, maps the dynamics of the system with a time-dependent configuration space, i.e. $[0, L(t)]$, to a system with a constant configuration space, i.e. $[0, L_0]$. The idea of transforming the problem with moving boundaries to an equivalent one with fixed boundaries was used previously by Munier *et al* [4], Razavy [22], Greenberger [6] and Seba [8].

We conclude this section by making a couple of remarks.

(a) Performing the canonical transformation (69) and (73), on the effective Hamiltonian (63) of Pereshogin and Pronin [1], we obtain the transformed effective Hamiltonian

$$H''_{\mathrm{eff}}(t) = \frac{L_0^2 p^2}{2ML(t)^2}. \tag{79}$$

This is the Hamiltonian of a particle with a time-dependent (effective) mass $\tilde{M}(t) = ML^2(t)/L_0^2$ which is confined between two walls positioned at $x = 0$ and L_0. The Schrödinger equation for $H''_{\mathrm{eff}}(t)$ can be easily solved, for the adiabatic approximation yields the exact result [23]. This means that the approach of Pereshogin and Pronin [1] leads to a Hamiltonian that is canonically equivalent to that of a free particle with a time-dependent mass. The eigenvalue problem for $H''_{\mathrm{eff}}(t)$ is also solved exactly and there is no need to appeal to perturbation theory.

(b) In the Schrödinger equation (77) for the transformed Hamiltonian (76), if one combines the term $\mathrm{i}\hbar\,\mathcal{U}(t)\dot{\mathcal{U}}(t)^\dagger = -\dot{L}(t)(xp + px)/(2L(t))$ with the time derivative, one obtains the 'covariant time derivative'

$$\nabla'_t := \frac{\partial}{\partial t} - \frac{\dot{L}}{2L}\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x\right). \tag{80}$$

The covariant time derivative ∇_t of equation (9) of Pereshogin and Pronin [1] differs from (80) by a minus sign in the second term on the right-hand side of (80). Indeed, as pointed out by one of the referees, a consistent treatment of the problem based on the method of Pereshogin and Pronin [1] shows that in fact (80) is the correct expression for the covariant time derivative. In order to see this, one must reconsider the definition of the operator $\hat{P} : L^2([0, L(t_1)]) \to L^2([0, L(t_2)])$ of Pereshogin and Pronin [1] which is used to define ∇_t. Pereshogin and Pronin determine $\hat{P}$ by requiring that its effect on the wavefunction (in the coordinate representation) be that of a dilatation. It is not difficult to see that $\hat{P} = \mathcal{U}(t)$ where $\mathcal{U}(t)$ is given by equation (69) with $\alpha(t) = \ln[L(t_2)/L(t_1)]$. Note that for $\psi(x, t_1) \in L^2([0, L(t_1)])$, $\hat{P}\psi_n(x, t_1) = \psi_n(x', t_1)$ where $x' = [L(t_2)/L(t_1)]x \in [0, L(t_2)]$. Hence, $\hat{P}\psi_n(x, t_1) \in L^2([0, L(t_2)])$, as required†. Setting $t_1 = t + \delta t$ and $t_2 = t$, one obtains the infinitesimal form of $\hat{P} : L^2([0, L(t + \delta t)]) \to L^2([0, L(t)])$ which is given by

$$\hat{P} = 1 - \delta t \, \frac{\dot{L}}{2L}\left(x\frac{\partial}{\partial x} + \frac{\partial}{\partial x}x\right). \tag{81}$$

Pereshogin and Pronin's expression for $\hat{P}$ differs from (81) in the sign of the second term on the right-hand side of (81). If one chooses the opposite sign, as Pereshogin and Pronin do, then $\hat{P}\psi_n(x, t_1) \notin L^2([0, L(t_2)])$, and the construction is inconsistent. If one uses expression (81) for $\hat{P}$ in Pereshogin and Pronin's analysis, one obtains the covariant time derivative (80) and the effective Hamiltonian

$$H'_{\rm eff} = \frac{p^2}{2M} - \frac{\dot{L}(t)}{2L(t)}\,(xp + px). \tag{82}$$

Note that again the relevant Hilbert space is $L^2([0, L(t)])$.

6. Adiabatic geometric phase due to moving boundaries

6.1. Adiabatic geometric phase for the Hamiltonian (64)

The Hamiltonian (64) is a special case of a parametric Hamiltonian of the form

$$H[R] = \frac{p^2}{2M} + \tilde{V}(x; R] \tag{83}$$

where

$$\tilde{V}(x; R] = \begin{cases} V(x, R] & \text{for} \quad x \in [0, L] \\ \infty & \text{for} \quad x \notin [0, L]. \end{cases} \tag{84}$$

$R = (L, R^1, \ldots, R^d)$ are real parameters and $R^1, \ldots, R^d$ may be viewed as a set of coupling constants occurring in the expression for V.

For the case of a free particle $V = 0$, and one can easily solve the eigenvalue equation for this Hamiltonian. The eigenvalues and eigenfunctions are given by

$$E_n = \frac{\hbar^2\pi^2 n^2}{2ML^2} \tag{85}$$

and

$$\phi_n = \sqrt{\frac{2}{L}}\,\sin\left(\frac{\pi n x}{L}\right)[\theta(x) - \theta(x - L)] \tag{86}$$

respectively. Since the eigenfunctions are real, one expects Berry's connection 1-form

$$A_n := \mathrm{i}\langle\phi_n|d|\phi_n\rangle \tag{87}$$

to vanish identically [24]. This is in fact the case for any real potential $V(x; t)$, because for a real potential the eigenfunctions ϕ_n may be chosen to be real. More specifically, one has

$$\phi_n(x; R] = f_n(x, R][\theta(x) - \theta(x - L)] \tag{88}$$

† Note that here $\hat{P}$ is an active transformation: $|\psi\rangle \to \hat{P}|\psi\rangle$. It can also be viewed as a passive transformation $\langle x| \to \langle x|\hat{P} =: \langle x'|$.

where f_n are real-valued functions depending on R and vanishing at $x = 0$ and L. A simple calculation shows that

$$A_n = \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x \left(f_n \sum_{a=0}^{n} \frac{\partial f_n}{\partial R^a}\, \mathrm{d}R^a\, [\theta(x) - \theta(L-x)]^2 + (f_n)^2[\theta(x) - \theta(L-x)]\delta(x-L)\,\mathrm{d}L \right)$$

$$= \tfrac{1}{2}\mathrm{i}\int_{0}^{L} \mathrm{d}x \left(\sum_{a=1}^{n} \frac{\partial (f_n)^2}{\partial R^a}\, \mathrm{d}R^a + \frac{\partial (f_n)^2}{\partial L}\,\mathrm{d}L + (f_n)^2[\theta(x) - \theta(L-x)]\delta(x-L)\,\mathrm{d}L \right)$$

$$= \tfrac{1}{2}\mathrm{i}\left[\left(\sum_{a=1}^{n} \mathrm{d}R^a \frac{\partial}{\partial R^a} + \mathrm{d}L\,\frac{\partial}{\partial L} \right) \left(\int_0^L f_n^2\,\mathrm{d}x \right) - \mathrm{d}L\,(f_n)^2 \Big|_{x=L} \right.$$

$$\left. + \,\mathrm{d}L\,[\theta(L) - \theta(0)](f_n)^2\Big|_{x=L} \right] \tag{89}$$

$$= 0. \tag{90}$$

The integral on the right-hand side of (89) is the norm of ϕ_n which is supposed to be one. Therefore, its derivatives vanish. The last two terms vanish, because $f_n|_{x=L} = 0$.

Equation (90) shows that the problem of the adiabatic geometric phase for the Hamiltonian (64) is trivial. This means that the cyclic adiabatic geometric phase angles $\gamma_n(T)$ vanish, and the non-cyclic adiabatic geometric phases $\Phi_n(t)$ only depend on the end points of the path traced by the parameters in the parameter space.

6.2. Adiabatic geometric phase for the canonically transformed Hamiltonian (76) with $V = 0$

The Hamiltonian (76) with $V = 0$ is obtained from the parametric Hamiltonian

$$H''[L, R] = \frac{L_0^2 p^2}{2ML^2} - \frac{R}{2ML^2}(xp + px) \tag{91}$$

by setting $L = L(t)$ and $R = R(t) = ML(t)\dot{L}(t)$. The Hilbert space of the system is $L^2([0, L_0])$.

We shall write H'' in the form

$$H''[L, R] = H_0[L] + \epsilon[R]\,h[L] \tag{92}$$

where

$$H_0[L] := \frac{L_0^2 p^2}{2ML^2} \qquad \epsilon[R] := R \qquad \text{and} \qquad h[L] := -\frac{xp + px}{2ML^2}. \tag{93}$$

The eigenvalues and eigenfunctions of H_0 are given by

$$E_n^{(0)} = \frac{\hbar^2\pi^2 n^2}{2ML^2} \qquad \text{and} \qquad \psi_n^{(0)}(x) = \langle x|n\rangle_0 = \sqrt{\frac{2}{L_0}}\,\sin\left(\frac{\pi n x}{L_0}\right) \tag{94}$$

respectively. Because $\psi_n^{(0)}(x)$ do not depend on R or L, $A_{mn}^{(0)} = 0$ and the Berry connection 1-form is given by equation (59). Using equations (30), (31) and (94) and performing the necessary algebra, we find

$$E_n^{(1)} = 0 \qquad \text{and} \qquad C_{mn}^{1} = \frac{4\mathrm{i}(-1)^{m+n} mn}{\hbar\pi^2(m^2 - n^2)^2} \qquad \text{for} \quad m \neq n. \tag{95}$$

Note that C_{mn}^1 do not involve L. Furthermore, in view of equations (34)–(37), E_n^ℓ will all be either zero or proportional to L^{-2}. Therefore, their ratios will also be independent of L. This in turn implies that all C_{mn}^ℓ should be independent of L. Hence $\mathrm{d}C_{mn}^\ell = 0$ for all ℓ. In view of equation (59) and $A_{mn}^{(0)} = 0$, this is sufficient to conclude that A_n is an exact 1-form and the geometric phase is trivial.

6.3. Adiabatic geometric phase for the effective Hamiltonians (63) and (82)

It is not difficult to see that the effective Hamiltonians (63) and (82) are special cases of a parametric Hamiltonian of the form

$$H_{\rm eff}[R] = \frac{p^2}{2M} + \frac{R}{2}\,(xp + px) \tag{96}$$

where $R \in \mathbb{R}$ is a real parameter. The effective Hamiltonians (63) and (82) are obtained from (96) by requiring R to change in time according to

$$R(t) = \pm\frac{\dot{L}(t)}{L(t)} \tag{97}$$

where the plus sign corresponds to the effective Hamiltonian (63) and the minus sign to the effective Hamiltonian (82). In the following we shall only treat the case of the effective Hamiltonian (63). The analogous results are obtained for the effective Hamiltonian (82) by changing the sign of R in the relevant equations.

It is well known, at least for the cases where the Hilbert space is $L^2(\mathbb{R})$, that a parametric Hamiltonian which has $\mathbb{R}$ as its parameter space cannot lead to a non-trivial geometric phase. This is simply because in this case Berry's connection 1-form A_n depends on a single variable $R \in \mathbb{R}$ and can be written as $\mathrm{d}F(R)$ where $F(R) = \int A_n(R)\,\mathrm{d}R$. This implies that both cyclic and non-cyclic adiabatic geometric phases are trivial. The same conclusion can also be reached for the cases when the Hilbert space is $L^2(\mathcal{M})$ where $\mathcal{M}$ is a fixed configuration space.

The configuration space of the effective Hamiltonian (63) is the interval $[0, L]$ which is variable. We can treat this case by identifying the Hilbert space $L^2([0, L])$ with $L^2_\mu(\mathbb{R})$ of equation (13) where $\mu = \theta(x) - \theta(x - L)$. In this way, it is clear that the expression for the Berry connection 1-form involves two parameters, namely R, which enters through the dependence of the eigenfunctions of $H_{\rm eff}[R]$ on R and L which enters through the dependence of the measure μ on L. Therefore, the above argument does not apply to $H_{\rm eff}[R]$.

Following Pereshogin and Pronin [1], we compute the eigenfunctions of $H_{\rm eff}[R]$ using perturbation theory. We shall write

$$H_{\rm eff} = H_0 + \epsilon h \tag{98}$$

where

$$H_0 = \frac{p^2}{2M} \qquad \epsilon = ML^2R \qquad \text{and} \qquad h = \frac{1}{2ML^2}(xp + px). \tag{99}$$

The eigenvalues and eigenfunctions of H_0 are given by

$$E_n^{(0)} = \frac{\hbar^2\pi^2n^2}{2ML^2} \qquad \text{and} \qquad \psi_n^{(0)}(x) = \langle x|n\rangle_0 = \sqrt{\frac{2}{L}}\,\sin\left(\frac{\pi n x}{L}\right). \tag{100}$$

Because the eigenfunctions $\psi_n^{(0)}(x)$ of H_0 are real, and $\psi_n^{(0)}(L) = 0$, we have $A_n^{(0)} = 0$. Furthermore, we can use equations (100), (30), (31) and (99), to calculate

$$A_{mn}^{(0)} = i_0\langle m|\frac{\mathrm{d}}{\mathrm{d}t}|n\rangle_0 = \frac{2\mathrm{i}mn(-1)^{m+n}\,\mathrm{d}L}{(m^2 - n^2)L} \qquad \text{for} \quad m \neq n \tag{101}$$

$$E_n^{(1)} = 0 \qquad \text{and} \qquad C_{mn}^1 = \frac{4\mathrm{i}(-1)^{m+n+1}mn}{\hbar\pi^2(m^2 - n^2)^2} \qquad \text{for} \quad m \neq n. \tag{102}$$

Again one can show that $E_n^{(\ell)}$ are all proportional to L^{-2}, and C_{mn}^{ℓ} are independent of the parameters. Hence, in view of equation (56), we have

$$\begin{aligned} A_n &= \left[2\sum_{m\neq n}\mathrm{Re}(C^1_{mn}A^{(0)}_{nm})\right]\epsilon+\cdots \\ &= \left[\left(\sum_{m\neq n}\frac{m^2n^2}{(m^2-n^2)^3}\right)\left(\frac{-16\,\mathrm{d}L}{\pi^2\hbar L}\right)\right]\epsilon+\cdots \\ &= \left(\sum_{m\neq n}\frac{m^2n^2}{(m^2-n^2)^3}\right)\left(\frac{-16MRL\,\mathrm{d}L}{\pi^2\hbar}\right)+\cdots \\ &= \left(\sum_{m\neq n}\frac{m^2n^2}{(m^2-n^2)^3}\right)\left(\frac{-16M\dot{L}\,\mathrm{d}L}{\pi^2\hbar}\right)+\cdots \end{aligned} \tag{103}$$

where '$\cdots$' denotes the terms which are either exact forms or of higher order in ϵ. Equation (103) coincides with the result of Pereshogin and Pronin [1]. Note that Γ^n_{tn} of Pereshogin and Pronin [1] is equal to $-\mathcal{A}_n = -A_n/\mathrm{d}t$.

It is worth mentioning that in view of equations (99), (97) and (101), both $\mathcal{A}^{(0)}_{mn}$ with $m\neq n$ and ϵ are proportional to $\dot{L}$. This shows that the choice made for the perturbation parameter ϵ is consistent with the adiabaticity of the evolution. In other words, the perturbation theory is valid for an adiabatic evolution of the system where $\dot{L}$ is very small.

7. Conclusion

In this paper we have addressed two problems. First, we presented a systematic perturbative calculation of Berry's connection 1-form and showed that Berry's phase due to a time-dependent perturbation is a second-order effect in the perturbation parameter. Next, we studied the quantum dynamics of a particle confined between moving walls and reconsidered the problem of the adiabatic geometric phase for this system.

We showed that using the conventional approach based on the Hamiltonian (64), this system does not involve any non-trivial adiabatic geometric phases. For the case of a free particle where $V=0$, transforming this system into a canonically equivalent one with fixed boundaries does not lead to non-trivial adiabatic geometric phases either. However, if one postulates a new effective Hamiltonian for the system, then in principle non-trivial geometric phases may arise even for the case of a free particle. For example, if one uses the effective Hamiltonian (63) or (82), one obtains a non-trivial adiabatic geometric phase. The effective Hamiltonian (63) turns out to be canonically equivalent to that of a free particle of variable mass which is confined to an infinite square well with fixed boundaries. The latter system can be solved exactly. In particular, one can show that it does not involve non-trivial adiabatic geometric phases. The occurrence of non-trivial adiabatic geometric phases for the effective Hamiltonian (63) has, therefore, its origin in the time-dependent canonical transformation relating the two systems [25].

We argued that a consistent treatment of the dynamics of a particle confined between a fixed and a moving boundary using the method of Pereshogin and Pronin [1] leads to an effective Hamiltonian which differs slightly from that obtained by Pereshogin and Pronin. The occurrence of non-trivial geometric phases for this effective Hamiltonian is a clear indication of the fact that the approach of Pereshogin and Pronin is not equivalent to the conventional approach. Since both approaches aim to describe the dynamics of the same physical system, an experimental investigation of their predictions can easily determine their validity.

Finally, we wish to remark that the dynamics of a massless particle confined between moving boundaries can also be treated by transforming the problem to an equivalent one with fixed boundaries [22]. The phenomenon of the geometric phase for such a particle has not been addressed in a satisfactory manner (see, however, [12]) though the geometric phases in optical systems have been thoroughly investigated (see, for example, the review [26]). The relativistic analogue of the adiabatic geometric phase has been discussed in [27]. The results of [27] may also be used to treat the case of massless particles satisfying the wave equation.

References

[1] Pereshogin P and Pronin P 1991 *Phys. Lett.* A **156** 12
[2] Berry M V 1984 *Proc. R. Soc.* A **392** 45
[3] Doescher S W and Rice M H 1969 *Am. J. Phys.* **37** 1246
[4] Munier A, Burgan J R, Feix M and Fijalkow E 1981 *J. Math. Phys.* **22** 1219
[5] Berry M V and Klein G 1984 *J. Phys. A: Math. Gen.* **17** 1805
[6] Greenberger D M 1988 *Physica* B **151** 374
[7] Pinder D N 1990 *Am. J. Phys.* **58** 54
[8] Seba P 1990 *Phys. Rev.* A **41** 2306
[9] Makowski A J and Dembinski S T 1991 *Phys. Lett.* A **154** 217
Makowski A J and Peplowski P 1992 *Phys. Lett.* A **163** 142
Makowski A J 1992 *J. Phys. A: Math. Gen.* **25** 3419
[10] Devoto A and Pomorisac B 1992 *J. Phys. A: Math. Gen.* **25** 241
[11] Dodonov V V, Klimov A B and Nikonov D E 1993 *J. Math. Phys.* **34** 3391
[12] Levy-Leblond J-M 1987 *Phys. Lett.* A **125** 441
[13] Aharonov Y and Anandan J 1987 *Phys. Rev. Lett.* **58** 1593
Anandan J and Aharonov Y 1988 *Phys. Rev.* D **38** 1863
[14] Mostafazadeh A 1997 *J. Math. Phys.* **38** 3489
[15] Mostafazadeh A 1998 *J. Phys. A: Math. Gen.* **31** 6495
[16] Born M and Fock V 1928 *Z. Phys.* **51** 165
Kato T 1950 *J. Phys. Soc. Japan* **5** 435
[17] Mostafazadeh A 1997 *Phys. Rev.* A **55** 1653
[18] Mostafazadeh A 1999 *J. Phys. A: Math. Gen.* **32** 8157
[19] de Polavieja G G and Sjöqvist E 1998 *Am. J. Phys.* **66** 431
[20] Bohm D 1989 *Quantum Theory* (New York: Dover)
[21] Bohm A 1993 *Quantum Mechanics: Foundations and Applications* 3rd edn (Berlin: Springer)
[22] Razavy M 1985 *Hadronic J.* **8** 153
Razavy M 1985 *Phys. Rev.* D **15** 307
Razavy M 1993 *Phys. Rev.* A **43** 3486
[23] Mostafazadeh A 1997 *Phys. Rev.* A **55** 4084
[24] Mostafazadeh A 1997 *Phys. Lett.* A **232** 395
[25] Giavarini G, Gozzi E, Rohrlich D and Thacker W D 1989 *Phys. Lett.* A **138** 235
[26] Bhandari R 1997 *Phys. Rep.* **281** 1
[27] Mostafazadeh A 1998 *J. Phys. A: Math. Gen.* **31** 7829